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ON BESSEL'S FUNCTIONS OF THE SECOND KIND.

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In the first volume of the Mathematische Annalen* (1869) Hankel has given an investigation of some of the properties of Bessel's functions. This paper is probably unknown or inaccessible to many who frequently have to make practical use of these functions in physics, so that I have thought it worth while to reproduce one of the important results contained in it† accompanying it with a few critical remarks.

Bessel's equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\, y = 0$$

is found (cf. Forsyth,‡ p. 158) to have as a particular solution, the following series, which is convergent for all values of x whatever value n may have:

$$J_n(x) = \left[\frac{x}{2}\right]^n \sum_{p=0}^{p=\infty} \frac{(-1)^p}{\Gamma(n+p+1)\,.\,\Gamma(p+1)} \left[\frac{x}{2}\right]^{2p}.$$

Here $\Gamma(n)$ is the Eulerian integral which reduces to $(n-1)!$ when n is a positive integer, and which is extended to negative values of n in any legitimate way, for instance by means of the formula $\Gamma(n+1) = n\,\Gamma(n)$.§

Thus defined, $J_n(x)$ is called a Bessel's function of x of the first kind of order n. Since the parameter n enters into the differential equation only through its square, $J_{-n}(x)$ will also be a solution of the equation, and in general the complete primitive of Bessel's equation will have the form $AJ_n(x) + BJ_{-n}(x)$. If however n is a real integer, the relation $J_n(x) = (-1)^n J_{-n}(x)$ holds true (Forsyth, p. 160); so that in this case the above solution of Bessel's equation is no longer general. Confining our attention then, as we may do without loss of generality, to *positive* integral values of n we have the problem: *To find in this case a solution of Bessel's equation linearly independent of* $J_n(x)$.

If the parameter of the equation were $n - \Delta n$ we should have as two independent solutions

$$J_{n-\Delta n}(x) \text{ and } J_{-n+\Delta n}(x).$$

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* p. 467. *Die Cylinderfunctionen erster und zweiter Art.*

† I have confined myself to § 1, although the following sections, especially the last two, are of the greatest interest to the physicist as well as to the student of the theory of functions.

‡ "A Treatise on Differential Equations." I shall frequently refer to this book by merely mentioning the author's name.

§ For the purpose of this note we may confine our attention to *real* values of n. This restriction however is in general undesirable and is not made by Hankel.

Any entire linear combination of these functions will accordingly also be a solution. Such a combination is

$$\frac{(-1)^n J_{-n+\Delta n}(x) - J_{n-\Delta n}(x)}{2\Delta n}.$$

Adding to the numerator of this fraction the null expression $J_n(x) - (-1)^n J_{-n}(x)$ will not affect it, so that we may write our solution

$$\frac{(-1)^{n+1}\,[J_{-n}(x) - J_{-n+\Delta n}(x)] + J_n(x) - J_{n-\Delta n}(x)}{2\Delta n}.$$

If, now, we let Δn approach zero, we obtain

$$K_n(x) = \tfrac{1}{2}\left[\frac{dJ_n(x)}{dn} + (-1)^{n+1}\frac{dJ_{-n}(x)}{dn}\right]$$

as a second solution of Bessel's equation *when n is a positive integer.* This function is called a Bessel's function of the second kind of order n.

Now we obtain at once by differentiation

$$\frac{dJ_n(x)}{dn} = \log\left[\frac{x}{2}\right].\ J_n(x) + \left[\frac{x}{2}\right]^n \sum_{p=0}^{p=\infty} \frac{(-1)^p}{\Gamma(p+1)}\left[\frac{x}{2}\right]^{2p} \frac{d[\Gamma(n+p-1)]^{-1}}{dn},$$

$$\frac{dJ_{-n}(x)}{dn} = -\log\left[\frac{x}{2}\right].\ J_{-n}(x) + \left[\frac{x}{2}\right]^{-n} \sum_{p=0}^{p=\infty} \frac{(-1)^p}{\Gamma(p+1)}\left[\frac{x}{2}\right]^{2p} \frac{d[\Gamma(-n+p+1)]^{-1}}{dn}.$$

In order to differentiate the reciprocals of $\Gamma(n+p+1)$ and $\Gamma(-n+p+1)$ with respect to n we will introduce a new function using a notation similar to that of Gauss, and write

$$\psi(x) = \frac{d\log\Gamma(x)}{dx}.$$

This function ψ will evidently have the property

$$\psi(x) + \frac{1}{x} = \psi(x+1);$$

so that if x is a positive integer,

$$\psi(x) = \frac{1}{x-1} + \frac{1}{x-2} + \ldots + \tfrac{1}{3} + \tfrac{1}{2} + 1 + \psi(1),$$

and $\psi(1)$ has approximately the value -0.5772. In terms of this function we shall have

$$\frac{d\Gamma(x)}{dx} = \Gamma(x).\ \psi(x),$$

and thus we get

$$\frac{dJ_n(x)}{dn} = \log\left[\frac{x}{2}\right] . J_n(x) - \left[\frac{x}{2}\right]^n \sum_{p=0}^{p=\infty} \frac{(-1)^p . \psi(n+p+1)}{\Gamma(n+p+1) . \Gamma(p+1)} \left[\frac{x}{2}\right]^{2p} .$$

The expression for the derivative of $J_{-n}(x)$ we might of course get in the same way, but this would introduce the factor $\frac{\psi(-n+p+1)}{\Gamma(-n+p+1)}$ which has for positive integral values of n and p the indeterminate form $\frac{\infty}{\infty}$ as long as $p \overset{=}{<} n-1$. We will therefore divide our series into two parts at this point, getting

$$\frac{dJ_{-n}(x)}{dn} = -\log\left[\frac{x}{2}\right] . J_{-n}(x) + \left[\frac{x}{2}\right]^{-n} \sum_{p=0}^{p=n-1} \frac{(-1)^p}{\Gamma(p+1)} \left[\frac{x}{2}\right] \quad \frac{d[\Gamma(-n+p+1)]^{-1}}{dn}$$
$$+ \left[\frac{x}{2}\right]^{-n} \sum_{p=n}^{p=\infty} \frac{(-1)^p \psi(-n+p+1)}{\Gamma(p+1) . \Gamma(-n+p+1)} \left[\frac{x}{2}\right]^{2p} .$$

In order to perform the differentiation in the cases which still remain resort must be had to a simple formula in the theory of the Γ function ; namely,

$$\Gamma(x) . \Gamma(1-x) . \sin x\pi = \pi ;$$

from which we get

$$\frac{1}{\Gamma(-n+p+1)} = \frac{1}{\pi} \sin(-n+p+1)\pi . \Gamma(n-p),$$

and therefore,

$$\frac{d}{dn}\left[\frac{1}{\Gamma(-n+p+1)}\right] = -\cos(-n+p+1)\pi . \Gamma(n-p)$$
$$-\frac{1}{\pi}\sin(-n+p+1)\pi . \Gamma(n-p) . \psi(n-p).$$

The value of this expression when n and p are positive integers and $p < n$ is

$$-(-1)^{-n+p+1} . \Gamma(n-p) = -(-1)^{n-p+1} . \Gamma(n-p).$$

Thus we have finally performed the differentiation indicated above. The series we get may however be written in a more convenient form if in the second (and infinite) part the new variable of summation $q = p - n$ be introduced. This gives us

$$\frac{dJ_{-n}(x)}{dn} = -\log\left[\frac{x}{2}\right] . J_{-n}(x) - (-1)^{n+1}\left[\frac{x}{2}\right]^{-n} \sum_{p=0}^{p=n-1} \frac{\Gamma(n-p)}{\Gamma(p+1)} \left[\frac{x}{2}\right]^{2p}$$
$$-(-1)^{n+1}\left[\frac{x}{2}\right]^{n} \sum_{q=0}^{q=\infty} \frac{(-1)^q \psi(q+1)}{\Gamma(n+q+1) . \Gamma(q+1)} \left[\frac{x}{2}\right]^{2q} .$$

We thus get finally as the development of our "Bessel's function of the second kind of order n" (where n of course represents a positive integer):

$$K_n(x) = \log\left[\frac{x}{2}\right] . J_n(x) - \tfrac{1}{2}\left[\frac{x}{2}\right]^{-n} \sum_{p=0}^{p=n-1} \frac{\Gamma(n-p)}{\Gamma(p+1)}\left[\frac{x}{2}\right]^{2p}$$

$$-\tfrac{1}{2}\left[\frac{x}{2}\right]^{n} \sum_{p=0}^{p=\infty} \frac{(-1)^p\,[\psi(n+p+1)+\psi(p+1)]}{\Gamma(n+p+1)\,.\,\Gamma(p+1)}\left[\frac{x}{2}\right]^{2p}.$$

I have now completed the account (somewhat fuller, and I hope on that account more easy to read, than the original) which I wished to give of § 1 of the paper of Hankel quoted above.*

In connection with this subject the following points may be mentioned:—

(1) That the series we have obtained for $\frac{dJ_n(x)}{dn}$ and $\frac{dJ_{-n}(x)}{dn}$ converge unconditionally may be seen at once by forming the quotient of the $(p+1)$ st to the pth term, as this quotient will be seen at once to become and remain as small as we please for any finite value of x if we take the number of terms p large enough. Nevertheless, we took one step above which still requires justification. We assumed, namely, that the derivatives with respect to n of the series

$$-\sum_{p=0}^{p=\infty} \frac{(-1)^p}{\Gamma(n+p+1)\,.\,\Gamma(p+1)}\left[\frac{x}{2}\right]^{2p} \text{ and } \sum_{p=0}^{p=\infty} \frac{(-1)^p}{\Gamma(-n+p+1)\,.\,\Gamma(p+1)}\left[\frac{x}{2}\right]^{2}$$

could be found by differentiating term by term. This, however, requires proof as there are many series (for instance, many trigonometric series) which cannot be so differentiated.

Now it is well known that a series may be differentiated term by term if the series obtained by performing the differentiation in this way converges *uniformly* for such values of the variable as we areconsidering. We obtained from the first of the above-written series by differentiating term by term the series:

$$-\sum_{p=0}^{p=\infty} \frac{(-1)^p\,\psi(n+p+1)}{\Gamma(n+p+1)\,.\,\Gamma(p+1)}\left[\frac{x}{2}\right]^{2p};$$

and the second of the above series gave us, if we neglect a finite number of terms which need not concern us here,

$$-(-1)^{n+1}\sum_{p=0}^{p=\infty} \frac{(-1)^p\,\psi(p+1)}{\Gamma(n+p+1)\,\Gamma(p+1)}\left[\frac{x}{2}\right]^{2p}.$$

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* The closing part of this section (quoted by Forsyth, p. 165, Ex. 1) of which the above may be considered a special case, would seem to be of less general interest.

These are the series which must be proved uniformly convergent for all positive values of n. This we can do at once by means of the theorem :*

When the series of the numerically greatest values assumed by the terms of the series in the interval in question converges, the series is uniformly convergent.

The terms will, however, all become numerically greatest for $n = 0$, and for this, as well as for every other value of n, the series have been seen to converge.

(2) The function $J_n(x)$ satisfies for all values of n the fundamental relations, which might be called with Gauss *relationes inter contiguas:*

$$nJ_n(x) = \tfrac{1}{2}\, x\, [J_{n+1}(x) + J_{n-1}(x)\,],$$

$$\frac{dJ_n(x)}{dx} = \tfrac{1}{2}\, [J_{n-1}(x) - J_{n+1}(x)\,].$$

In fact this seems to be the reason† why the coefficient of x^n in the development of $J_n(x)$ is taken as $[2^n \Gamma(n+1)\,]^{-1}$; for if it were given a simpler value, these relations would not be satisfied.

It is accordingly of the highest practical importance, that the new function $K_n(x)$ should also satisfy similar relations; and from our method of introducing this function it is evident that it will do so. For we have

$$(n - \Delta n)\, J_{n-\Delta n}(x) = \tfrac{1}{2}\, x\, [J_{n+1-\Delta n}(x) + J_{n-1-\Delta n}(x)\,],$$

$$-\,(n - \Delta n)\, J_{-n+\Delta n}(x) = \tfrac{1}{2}\, x\, [J_{-n+1+\Delta n}(x) + J_{-n-1+\Delta n}(x)\,].$$

From these equations we get

$$(n - \Delta n)\, \frac{J_{n-\Delta n}(x) - (-1)^n J_{-n+\Delta n}(x)}{2\Delta n} = \tfrac{1}{2}\, x \left[\frac{J_{n+1-\Delta n}(x) - (-1)^{n+1} J_{-n-1+\Delta n}(x)}{2\Delta n} \right.$$
$$\left. + \frac{J_{n-1-\Delta n}(x) - (-1)^{n-1} J_{-n+1+\Delta n}(x)}{2\Delta n} \right].$$

Or, in the limit, when $\Delta n = 0$,

$$nK_n(x) = \tfrac{1}{2}\, x\, [K_{n+1}(x) + K_{n-1}(x)\,].$$

The other fundamental relation can be proved in exactly the same way.

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* cf. Harnack: Die Elemente der Differential- und Integralrechnung. Leipzig, 1881. Translated by G. L. Cathcart, 1891. Book III, Chap. IV, should be consulted.

† I mean the reason which is of weight with practical physicists. Mathematically there are several other reasons for this choice, which, however, have nothing to do with the definition of $J_n(x)$ as a solution of a differential equation.

(3) We may, however, find another second solution of Bessel's equation satisfying these relations by adding to $K_n(x)$ defined as above $A J_n(x)$, where A is a function neither of n nor of x.* For practical purposes it will be advisable to simplify $K_n(x)$ as much as possible by this means. The first term of $K_n(x)$ is $\log(x/2) \,.\, J_n(x)$ which may be reduced in this way to $\log x \,.\, J_n(x)$. We have already noticed that when x is a positive integer,

$$\psi(x) = \frac{1}{x-1} + \frac{1}{x-2} + \dots + \tfrac{1}{3} + \tfrac{1}{2} + 1 + \psi(1).$$

The constant $\psi(1)$ may, however, be omitted from all our ψ's; since it is readily seen that by doing so we merely add $\psi(1) \,.\, J_n(x)$ to $K_n(x)$. We can then write our new solution, which I will denote by $\{K_n(x)\}$, in the following form, hardly less adapted to numerical calculation than the ordinary development of $J_n(x)$:

$$\{K_n(x)\} = \log x \, J_n(x) - \tfrac{1}{2}\left[\frac{x}{2}\right]^{-n} \sum_{p=0}^{p=n-1} \frac{(n-p-1)!}{p!}\left[\frac{x}{2}\right]^{2p}$$

$$-\tfrac{1}{2}\left[\frac{x}{2}\right]^{n} \sum_{p=0}^{p=\infty} \frac{(-1)^p}{(n+p)!\,p!}\left[1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots + \frac{1}{p} + 1 + \tfrac{1}{2} + \dots + \frac{1}{n+p}\right]\left[\frac{x}{2}\right]^{2p}.$$

(4) Finally it remains to mention the other forms in which the solution $K_n(x)$ is usually found,† namely, a series proceeding either wholly or in part according to Bessel's functions of the first kind $J_p(x)$, the positive integer p being the variable of summation. On the contrary Hankel's series is merely a power series‡ except for the necessary logarithmic terms. Hankel's expression for $K_n(x)$ is therefore the most elementary in form, and I think no one who is familiar with the methods of obtaining the other expressions for $K_n(x)$ will deny that Hankel's expression is also far more easily found. Moreover, his has the advantage, as all students of the modern theory of linear differential equations know, of being a general method, applicable to the large class of cases where logarithmic terms occur in the solution. Why should the working physicist not have the advantage of this simple and in the highest degree practical theory ?

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* Of course we might also multiply $K_n(x)$ by any factor not involving n or x.

† *C. Neumann:* Theorie der Besselschen Functionen. 1867.
Lommel: Studien über die Besselschen Functionen. 1868. Also *Forsyth,* § 105.

‡ The function $J_n(x)$ which occurs in the above expression for $K_n(x)$ could of course be expressed as a power series too, if desired. The above form, however, will probably be as convenient in most cases, as $K_n(x)$ is rarely used except when $J_n(x)$ has also to be computed.